By Theorem 2, throughput θ_i increases at price p if

$$\frac{\alpha_i p}{\beta_i \phi} < \frac{\sum_{j\in\mathcal{N}} \alpha_j \theta_j}{\mu + \sum_{k\in\mathcal{N}} \beta_k \theta_k}. \tag{8}$$

We set the system capacity to be $\mu = 1$ and consider a set of 9 types of CPs with values of (α_i, β_i) chosen from $\{1, 3, 5\}$.

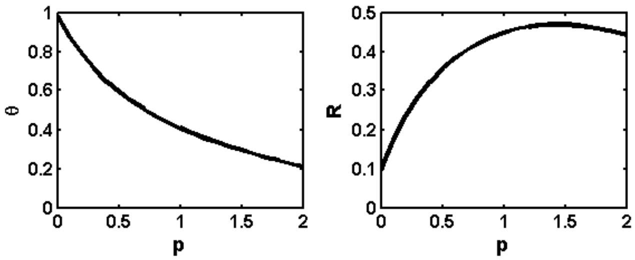

Figure 4: Aggregate throughput θ and ISP's revenue R.

Figure 4 plots the aggregate throughput θ (left) and the ISP's revenue R (right) as a function of price p that varies along the x-axis. We observe that the aggregate throughput decreases with the price as indicated by Theorem 2; however, the revenue $R = p\theta$ depends on both the price and aggregate throughput and shows a single-peak pattern.

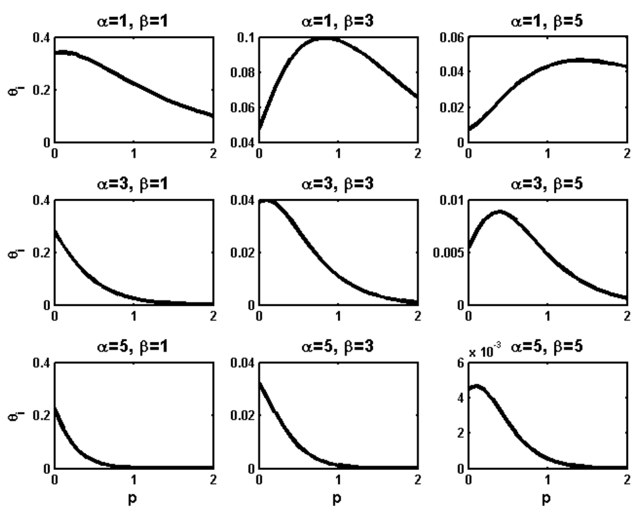

Figure 5: Throughput θ_i of different CPs.

Figure 5 shows the throughput θ_i of the 9 individual CPs as a function of the price p in each sub-figure, respectively. In general, the throughput is low with large values of α_i and β_i (the lower and right sub-figures), because the user population is more sensitive to price and the per user throughput is more sensitive to congestion. By Theorem 2, when p increases, ϕ decreases, and therefore $(\alpha_i/\beta_i)(p/\phi)$ increases. As indicated by condition (8), each θ_i decreases with p eventually; however, when p is small, we observe that the CPs with a small ratio of α_i/β_i (the upper and right sub-figures) demonstrate an increasing trend in throughput. Intuitively, for these CPs, the increase in the per user throughput λ_i is much higher than the decrease in the user population m_i so that the aggregate throughput θ_i could still increase.

Regulatory Implications: Under the existing one-sided pricing, an increase in the access ISP's price reduces the user demand (by Assumption 2), and consequently reduces the system utilization and the total system throughput (by Theorem 2). Regulators might want to regulate the price of an access ISP if its high price induces low utilization and drives the total system throughput too low (shown in Figure 4). However, as the existing wireless capacities of many carriers are often under-provisioned and highly-loaded, and regulations will further limit the ISPs' profit margin, we do not see the need for price regulation under the status quo.

4. SUBSIDIZATION COMPETITION

As the access ISPs can neither differentiate services nor charge CPs in the one-sided pricing model in Figure 3, the limited profit margin does not provide enough investment incentive for them to expand capacities. Neither do the CPs have any means to express preferences and improve their utilities. In this section, we propose to create a feedback channel for the CPs to influence the system by allowing them to voluntarily subsidize the usage-based fees for their users.

4.1 The Subsidization Competition Model

We denote $q \geq 0$ as a subsidization policy that limits the maximum subsidy allowed. We denote $s_i \in [0, q]$ as the per-unit usage subsidy provided by CP i for its content traffic for users. We denote $\mathbf{s}$ as the vector of subsidies of the CPs. We extend the definition of CP i's utility as $U_i \triangleq (v_i - s_i)\theta_i$.

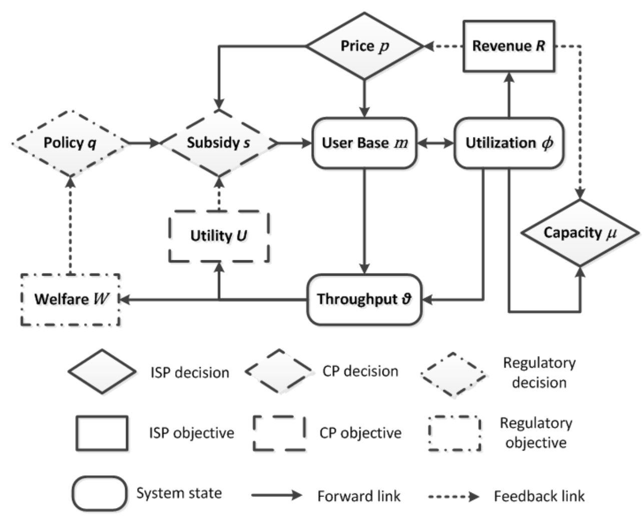

Figure 6: One-sided ISP pricing with CP subsidization.

Figure 6 illustrates the extended model where subsidization from CPs is allowed. In this model, each CP i could strategically choose its subsidy s_i to influence its throughput θ_i via its user population m_i so as to optimize its utility U_i. Regulators can also use any welfare metric W to determine the desirable upper-bound q of subsidization for the industry.

Under price p and subsidies $\mathbf{s}$, user population m_i satisfies $m_i(t_i) = m_i(p - s_i)$. Given any fixed ISP decision (p, μ), we denote $\phi(\mathbf{s}) \triangleq \phi(\mathbf{m}(p, \mathbf{s}), \mu)$ as the system utilization and define $\theta_i(\mathbf{s}) = m_i(p - s_i)\lambda_i(\phi(\mathbf{s}))$ as CP i's throughput.

Lemma 3. *For any $s'_i > s_i$, let $\mathbf{s}' = (s'_i, \mathbf{s}_{-i})$ and $\mathbf{s} = (s_i, \mathbf{s}_{-i})$ for some fixed strategy profile $\mathbf{s}_{-i}$. For any price p, $\phi(\mathbf{s}') \geq \phi(\mathbf{s})$, $\theta_i(\mathbf{s}') \geq \theta_i(\mathbf{s})$ and $\theta_j(\mathbf{s}') \leq \theta_j(\mathbf{s})$, $\forall j \neq i$.*

Lemma 3 states that if a CP i unilaterally increases its subsidy s_i, its throughput θ_i and the system utilization ϕ might increase; however, the throughput of any other CP cannot increase. It shows that the subsidization mechanism creates a competitive game among the CPs, where each CP i could use subsidy s_i to maximize its utility U_i. Notice that besides the policy constraint q, each CP's optimal subsidy strategy depends on all other CPs' strategies as well as the ISP's price p. In this section, we study the policy implications of subsidization and therefore, we often assume a fixed price p for the access ISP. This setting corresponds to a competitive access market where the ISPs cannot easily manipulate prices, or a case where the ISP's price is regulated. We will study an ISP's pricing strategy, its revenue and the policy implication on the system welfare in the next section.

Definition 3 (NASH EQUILIBRIUM). *Given any fixed ISP price p and policy q, a strategy profile $\mathbf{s}$ is a* Nash equilibrium *if each s_i solves CP i's utility maximization problem:*

$$\begin{aligned} \max \quad & U_i(s_i;\mathbf{s}_{-i}) = (v_i - s_i)\theta_i(\mathbf{s}) \\ \text{subject to} \quad & 0 \le s_i \le q. \end{aligned}$$

We characterize the Nash equilibrium of the subsidization competition game in the following theorem.

Theorem 3 (CHARACTERIZATION). *For any fixed p and q, a strategy profile $\mathbf{s}$ is a Nash equilibrium only if*

$$s_i = \min\{\tau_i(\mathbf{s}), q\}, \quad \forall\, i \in \mathcal{N},$$

where each τ_i denotes a threshold for CP i, defined by

$$\tau_i(\mathbf{s}) \triangleq (v_i - s_i)\,\epsilon_{s_i}^{m_i}\left(1 + \epsilon_{\phi}^{\lambda_i}\epsilon_{m_i}^{\phi}\right). \tag{9}$$

In particular, $v_i \le (\partial\theta_i/\partial s_i)^{-1}\theta_i$ holds if $s_i = 0$. Moreover, if $U_i(\mathbf{s}) = U_i(s_i, \mathbf{s}_{-i})$ is concave in s_i for all $i \in \mathcal{N}$, then the above conditions are also sufficient.

Theorem 3 characterizes the Nash equilibrium in two ways. First, it indicates that any CP i does not subsidize if its profitability v_i is lower than $(\partial\theta_i/\partial s_i)^{-1}\theta_i$. Second, if the policy constraint is not tight, i.e., $s_i < q$, the equilibrium subsidy s_i can be characterized by its profit margin $v_i - s_i$ and the elasticity metrics $\epsilon_{s_i}^{m_i}$, $\epsilon_{\phi}^{\lambda_i}$ and $\epsilon_{m_i}^{\phi}$. Equation (9) can also be written as $\tau_i(\mathbf{s}) = (v_i - s_i)\epsilon_{s_i}^{\theta_i} = (v_i - s_i)\epsilon_{s_i}^{m_i}\left(1 + \epsilon_{m_i}^{\lambda_i}\right)$. This characterization also implies that a CP would subsidize its users more if 1) its traffic profitability v_i (and therefore its profit margin $v_i - s_i$) increases, or 2) its throughput elasticity $\epsilon_{s_i}^{\theta_i}$ or user demand elasticity of subsidy $\epsilon_{s_i}^{m_i}$ increases.

By Assumption 1 and 2, each $U_i(\mathbf{s})$ is differentiable in $\mathbf{s}$. For any strategy profile $\mathbf{s}$, we define $\mathbf{u}(\mathbf{s}) = \{u_i(\mathbf{s}) : i \in \mathcal{N}\}$ as the vector of marginal utilities, where each $u_i(\mathbf{s})$ denotes the marginal utility of CP i, defined as $u_i(\mathbf{s}) \triangleq \partial U_i(\mathbf{s})/\partial s_i$. Next, we characterize the uniqueness of Nash equilibrium based on a condition on the marginal utilities $\mathbf{u}(\mathbf{s})$.

Theorem 4 (UNIQUENESS). *For any fixed price p and policy q, if for any distinct pair of feasible strategy profiles $\mathbf{s}' \neq \mathbf{s}$, there always exist a CP $i \in \mathcal{N}$ such that*

$$(s_i' - s_i)\left(u_i(\mathbf{s}') - u_i(\mathbf{s})\right) < 0, \tag{10}$$

then there always exists a unique Nash equilibrium.

Technically, the above sufficient condition for uniqueness requires $-\mathbf{u}$ to be a *P-function* [31]. Notice that this uniqueness property holds for any subset $\mathcal{Q} \subset [0, q]^{|\mathcal{N}|}$ of the strategy space if condition (10) holds in the domain $\mathcal{Q}$.

4.2 Dynamics of Subsidies in Equilibrium

Since a Nash equilibrium is defined under a fixed price p and policy q by Definition (3), we use $\mathbf{s} = \mathbf{s}(p, q)$ to indicate a Nash equilibrium under p and q. Equation (9) in Theorem 3 hinted that a CP with higher profitability subsidizes users more in an equilibrium. We formally show this as follows.

Theorem 5 (PROFITABILITY EFFECT ON SUBSIDY). *If a CP i's profitability increases from v_i to $\hat{v}_i$ unilaterally, and under the condition (10) of Theorem 4, $\mathbf{s}$ and $\hat{\mathbf{s}}$ are the corresponding Nash equilibria, then we must have $\hat{s}_i \ge s_i$.*

If a CP's profitability increases, Theorem 5 tells that it will increase its subsidy in equilibrium, and hints that its throughput will increase by Lemma 3. Through subsidization, CPs can respond to their profitability and influence throughput.

After analyzing the impact of CPs' profitability, we study how an equilibrium $\mathbf{s}(p, q)$ is influenced by the price p and policy q. We define $\mathcal{N}_-$ and $\mathcal{N}_+$ as the set of CPs whose subsidies are 0 and q, respectively, and $\widetilde{\mathcal{N}} = \mathcal{N}\backslash(\mathcal{N}_- \cup \mathcal{N}_+)$ as the remaining set of CPs whose subsidies are strictly positive and less than q. We define $\tilde{\mathbf{s}}$ and $\tilde{\mathbf{u}}$ as the subsidies and marginal utilities of the set $\widetilde{\mathcal{N}}$ of CPs.

Theorem 6 (EQUILIBRIUM DYNAMICS). *If a Nash equilibrium $\mathbf{s}$ satisfies 1) $i \in \widetilde{\mathcal{N}}$ if $u_i(\mathbf{s}) = 0$, and 2) the condition (10) for a neighborhood of $\mathbb{R}^{|\mathcal{N}|}$ centered at $\mathbf{s}$ in the strategy space, then in a neighborhood of (p, q), a unique Nash equilibrium is a differentiable function $\mathbf{s}(p, q)$, satisfying*

$$\frac{\partial s_i}{\partial q} = \begin{cases} 0 & \text{if } i \in \mathcal{N}_-; \\ 1 & \text{if } i \in \mathcal{N}_+; \\ -\sum\limits_{k\in\widetilde{\mathcal{N}}} \psi_{ik} \sum\limits_{j\in\mathcal{N}_+} \dfrac{\partial u_k}{\partial s_j}, & \text{if } i \in \widetilde{\mathcal{N}}, \end{cases} \tag{11}$$

$$\text{and} \quad \frac{\partial s_i}{\partial p} = \begin{cases} 0 & \text{if } i \notin \widetilde{\mathcal{N}}; \\ -\sum\limits_{k\in\widetilde{\mathcal{N}}} \psi_{ik} \dfrac{\partial u_k}{\partial p}, & \text{if } i \in \widetilde{\mathcal{N}}, \end{cases} \tag{12}$$

where $\Psi = \{\psi_{ij}\} \triangleq (\nabla_{\tilde{\mathbf{s}}}\tilde{\mathbf{u}})^{-1}$, i.e., ψ_{ij} is at the ith row and jth column of the inverse of the Jacobian matrix $\nabla_{\tilde{\mathbf{s}}}\tilde{\mathbf{u}}$.

Theorem 6 states that a marginal change in the ISP's price or the regulatory policy will not affect the behavior of the set $\mathcal{N}_-$ and $\mathcal{N}_+$ of CPs: the CPs who do not subsidize remain the same and the CPs who subsidize the amount q keep subsidizing at the maximum level as q increases. When p changes, the set $\widetilde{\mathcal{N}}$ of CPs readjust subsidies among themselves in a new equilibrium such that their marginal utilities remain zero in (12); when q changes, since the set $\mathcal{N}_+$ of CPs increase subsidies accordingly, their impacts have to be counted in the new equilibrium in (11). By applying the KKT condition to Definition 3, we know $u_i(\mathbf{s}) = 0$ for all $i \in \widetilde{\mathcal{N}}$, and the first condition of Theorem 6 is a bit stronger which guarantees that the equilibrium $\mathbf{s}$ is *regular*, i.e., locally differentiable. The second condition only assumes (10) locally, which guarantees the local uniqueness of equilibrium in a neighborhood centered at $\mathbf{s}$ by Theorem 4.

Corollary 1 (DEREGULATION). *Under a fixed ISP price p and the conditions of Theorem 6, in a neighborhood of q, we can express $\phi(q)$ and $R(q)$ as the system utilization and the ISP's revenue under the Nash equilibrium $\mathbf{s}(q)$. If $\mathbf{u}$ is off-diagonally monotone, i.e., $\partial u_i(\mathbf{s})/\partial s_j \ge 0$ for all $i \neq j$,*

$$\frac{\partial \phi}{\partial q} \ge 0, \quad \frac{\partial R}{\partial q} \ge 0 \quad \text{and} \quad \frac{\partial s_i}{\partial q} \ge 0, \quad \forall i \in \mathcal{N}.$$

The off-diagonally monotone property makes $-\mathbf{u}$ to be a Leontief type [18] of P-matrix, which guarantees the stability of a macroeconomic system due to Wassily Leontief [20]. Intuitively, it assumes although the utility of a CP i decreases when other CPs increase their subsidies, its marginal benefit of subsidizing its own users, i.e., u_i, increases. Corollary 1 states that under this stability condition, when allowed to subsidize more, CPs will increase subsidies, which will lead to an increase in the system utilization and the ISP's revenue.

Regulatory Implications: Under a competitive or price regulated access market, deregulation of subsidization encourages CPs to subsidize users (by Theorem 5 and 6), and consequently increases the system utilization and the ISPs' revenue (by Corollary 1). CPs' user populations $\mathbf{m}$ increase because the usage charges become cheaper after subsidization (by Assumption 2), but the induced higher utilization (by Theorem 1) might cause congestion. The total system throughput also increases (implied by Corollary 1 as ϕ increases), while the throughput of certain CPs might decrease. However, this is caused by the negative network externality, i.e., the physics of congestion of shared capacity. Although it might be unfavorable for the CPs whose traffic is more congestion sensitive in the short term, as the ISPs improve profit margins from higher utilization, they will have more incentives to expand capacities so as to accommodate more traffic and relieve congestion in the long term.

5. ISP REVENUE AND SYSTEM WELFARE

Although the policy q, the price p and the subsidies $\mathbf{s}$ all impact the system, these decisions are not made simultaneously and independently. The system is fundamentally driven by a regulatory policy q, under which the ISP determines its price $p(q)$ and then the CPs respond with strategies $\mathbf{s}(p, q)$. Theorem 6 characterizes the dynamics of $\mathbf{s}(p, q)$ and Corollary 1 characterizes the impact of policy q when p is fixed. In practice, ISPs often use price to optimize revenue, and by Theorem 6 the subsidies $\mathbf{s}$ are also influenced by p. In this section, we study the impact of the ISP's price on its revenue and the impact of policy on the system welfare when the response of ISP's pricing is taken into account.

5.1 Impact of ISP's Pricing on Its Revenue

Under a fixed policy q, by Theorem 6, we can write $\mathbf{s}(p)$ as the induced subsidies under price p. Thus, the induced system utilization is $\phi(\mathbf{s}(p))$ and the ISP's induced revenue is $R(p) = p\sum_{i\in\mathcal{N}}\theta_i(p) = p\sum_{i\in\mathcal{N}} m_i(p - s_i(p))\lambda_i(\phi(\mathbf{s}(p)))$.

Theorem 7 (MARGINAL REVENUE). *If the ISP's revenue is differentiable at p, then its marginal revenue is*

$$\frac{dR(p)}{dp} = \sum_{i\in\mathcal{N}}\theta_i + \Upsilon\sum_{i\in\mathcal{N}}\epsilon_p^{m_i}\theta_i, \tag{13}$$

where Υ and the m_i-elasticity of price p satisfy

$$\Upsilon = 1 + \sum_{j\in\mathcal{N}}\epsilon_{m_j}^{\lambda_j} \quad \text{and} \quad \epsilon_p^{m_i} = \frac{p}{m_i}\frac{dm_i}{dt_i}\left(1 - \frac{\partial s_i}{\partial p}\right).$$

Theorem 7 characterizes the change in revenue as the price varies and isolates the effect of subsidization into the elasticities $\epsilon_p^{m_i}$, where $\partial s_i/\partial p$ plays a role. It also generalizes the case of one-sided pricing, i.e., $\partial s_i/\partial p = 0$ for all $i \in \mathcal{N}$. The second term in (13) measures the price's impact on the aggregate throughput, i.e., $p(\partial\theta/\partial p)$, which can be factorized by Υ, a parameter determined by the physical model in Figure 2, because $\epsilon_{m_j}^{\lambda_j}$ can be decomposed as $\epsilon_{m_j}^{\phi}\epsilon_{\phi}^{\lambda_j}$ and

$$\epsilon_{m_j}^{\phi}\epsilon_{\phi}^{\lambda_j} = \frac{m_j}{\lambda_j}\frac{d\lambda_j}{d\phi}\frac{\partial\phi}{\partial m_j} = m_j\frac{d\lambda_j}{d\phi}\left(\frac{dg}{d\phi}\right)^{-1}. \tag{14}$$

5.2 Impact of Regulatory Policy on Welfare

By Theorem 7, we understand how the ISP's pricing affects its revenue. We assume that the ISP would adapt its price p as a function of a given policy q. The next theorem captures the impact of the policy q on the system states, i.e., ϕ, m_i and θ_i, where the responses of both the ISP's price $p(q)$ and the CPs' subsidies $\mathbf{s}(p, q)$ are taken into account.

Theorem 8 (POLICY EFFECT). *If the ISP's price is a differentiable function $p(q)$ of policy q and $\mathbf{s}$ is a Nash equilibrium of $p(q)$ and q, under the conditions of Theorem 6, in a neighborhood of q, we can express the Nash equilibrium as $\mathbf{s}(q) \triangleq \mathbf{s}(p(q), q)$. The corresponding system utilization $\phi(q)$, user population $m_i(q)$ and throughput $\lambda_i(q)$ satisfy*

$$\frac{dm_i}{dq} = \frac{dm_i}{dt_i}\frac{dt_i}{dq} = \frac{dm_i}{dt_i}\left(\left(1 - \frac{\partial s_i}{\partial p}\right)\frac{dp}{dq} - \frac{\partial s_i}{\partial q}\right), \tag{15}$$

$$\frac{d\phi}{dq} = \left(\frac{dg}{d\phi}\right)^{-1}\sum_{i\in\mathcal{N}}\frac{dm_i}{dq}\lambda_i \quad \text{and} \quad \frac{d\lambda_i}{dq} = \frac{d\lambda_i}{d\phi}\frac{d\phi}{dq}. \tag{16}$$

Any CP i's throughput θ_i increases with q if and only if

$$\epsilon_{t_i}^{m_i}\epsilon_q^{t_i}/\epsilon_{\phi}^{\lambda_i} < -\epsilon_q^{\phi}. \tag{17}$$

Theorem 8 shows that the policy effect on utilization $d\phi/dq$ in (16) and the condition (17) for throughput θ_i have similar forms as the price effect $d\phi/dp$ in (5) and the condition (7). This is because the policy effect is carried out by its impact on the user populations, i.e., $\epsilon_q^{m_i} = \epsilon_{t_i}^{m_i}\epsilon_q^{t_i}$, via its impact on the price and subsidies. By Assumption 1 and 2, both $\epsilon_{\phi}^{\lambda_i}$ and $\epsilon_{t_i}^{m_i}$ are negative. Inequality (17) tells that a CP's throughput decreases if any only if $(-\epsilon_{t_i}^{m_i})(-\epsilon_q^{t_i}) < -\epsilon_{\phi}^{\lambda_i}\epsilon_q^{\phi}$, which implies that the subsidy could help increase throughput via $(-\epsilon_q^{t_i})$; however, $(-\epsilon_{t_i}^{m_i})$ could be decreased due to the ISP's increasing price, which will reduce the CP's throughput.

Corollary 2 (POLICY IMPACT ON WELFARE). *Under the conditions of Theorem 8, let $W(q) \triangleq \sum_{i\in\mathcal{N}}\theta_i(q)v_i$ define the system welfare. Suppose $d\phi/dq$ in (16) is positive, then the marginal welfare dW/dq is positive if and only if*

$$\sum_{i\in\mathcal{N}}\frac{w_i}{\sum_{k\in\mathcal{N}}w_k}v_i > \sum_{i\in\mathcal{N}}-\epsilon_{m_i}^{\lambda_i}v_i, \quad \text{where} \quad w_i \triangleq \lambda_i\frac{dm_i}{dq}.$$

We measure the welfare by $W = \sum_{i\in\mathcal{N}}\theta_i v_i$ for two reasons. First, it internalizes the subsidy transfer from CPs to ISP. Second, because CP profits are often positively correlated to their values to users, it also serves an estimate for user welfare. In Corollary 2, $w_i v_i$ can be interpreted as the increase in welfare due to the policy's impact on the user population m_i, and therefore, the left side of the inequality represents the normalized increase in welfare due to the changes in the user populations. The right side of the inequality represents the normalized decrease in welfare due to the policy's impact on each average throughput λ_i via m_i. Corollary 2 states that the welfare increases if and only if the increasing component is larger than the decreasing component. Notice that the decreasing component only depends on the physical characteristics (14) of the

CPs, while the weight $w_i / \sum_{j\in\mathcal{N}} w_j$ for each v_i tends to be large if v_i is large, because profitable CPs will have stronger tendencies to subsidize their users so as to attract a larger population m_i.

To understand the policy effect more intuitively, we perform a numerical evaluation as follows. We use the same setting $m_i(t_i) = e^{-\alpha_i t_i}$, $\lambda_i(\phi) = e^{-\beta_i \phi}$ and $\Phi(\theta,\mu) = \theta/\mu$ as in Section 3. We model 8 types of CPs with $\alpha_i, \beta_i \in \{2,5\}$ and $v_i \in \{0.5, 1\}$. By Lemma 2, each CP represents the aggregation of a group of CPs with similar characteristics of traffic and user demand. In each of the following figures, we vary the policy q at 5 levels from 0 to 2.0 and vary the ISP's price p from 0 to 2 along the x-axis. When $q = 0$, the figures show the baseline case where subsidization is not allowed.

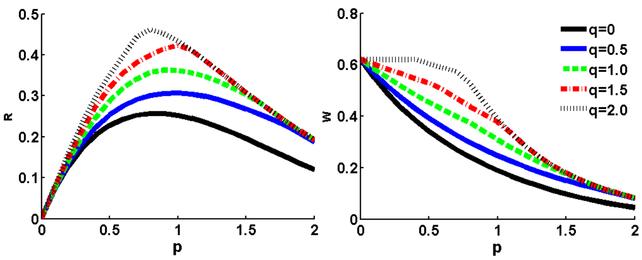

Figure 7: ISP's Revenue R and System Welfare W.

Figure 7 plots the ISP's revenue R (left) and the system welfare W (right) when the price p varies along the x-axis. We observe that under any fixed price p, both the ISP's revenue and the system welfare are higher when the CPs' subsidization is less regulated, i.e., q is large. However, if the deregulation of subsidization will trigger a higher ISP price, it also shows that the system welfare will decrease with the price p under any fixed policy q.

Figures 8 and 9 plot the individual subsidy s_i and throughput θ_i of the 8 types of CPs under equilibrium, respectively[2]. In Figure 8, we observe that the CPs that have a higher profitability, i.e., $v_i = 1$ in the lower 4 sub-figures, or a higher demand elasticity, i.e., $\alpha_i = 5$ in the right 4 sub-figures, provide much higher subsidies compared to their counterparts. We also observe that when the price p is small, except for the two CPs with $\alpha_i = 2$ and $v_i = 0.5$, most CPs want to subsidize at the maximum level q constrained by the policy; however, when p increases, subsidies may stay flat and then decrease due to the decrease in profit margin. By comparing Figure 8 with Figure 7, we observe that when $q = 2$, the ISP maximizes its revenue by setting p a bit less than 1, where the subsidies from the CPs are kept at a high level. In Figure 9, we observe that the CPs with higher profitability, i.e., $v_i = 1$, or lower congestion elasticity, i.e., $\beta_i = 2$, achieve higher throughput compared to their counterparts. When comparing the throughput with that of the baseline, i.e., $q = 0$, we observe that the CPs with high profitability achieve higher throughput, with the only exception for the case $(\alpha_i, \beta_i, v_i) = (2, 5, 1)$ when p is small. As p is small, the system attracts large user demand and is relatively congested. Because this CP is more congestion sensitive, i.e., β_i is large, and by Corollary 1, subsidization will further increase the system utilization, its throughput will be reduced. By comparing Figure 9 with Figure 7, we observe that with higher throughput for the CPs of higher profitability, the system welfare increases when the policy q is relaxed.

Regulatory Implications: In a deregulated market where CPs are allowed to subsidize their users, highly profitable CPs will provide greater subsidies (by Theorem 5) and obtain more users (by Assumption 2), achieving higher throughput (by Theorem 1) and leading to a higher system welfare (by Corollary 2). However, deregulation of subsidization might trigger an increase in the ISP's price, leading to a decrease in the system welfare (shown in Figure 7). High ISP prices also discourage user demand (by Assumption 2) and consequently decrease their throughput (shown in Figure 9). Our results suggest that the subsidization competition will increase the ISP's revenue and system welfare (by Corollary 1); however, price regulations might be needed if the access market is not competitive and the ISP's price is set too high.

[2]Figures of population m_i and utility U_i can also be found in [27].

6. DISCUSSIONS

In this section, we discuss related issues of subsidization, its connections to other recent developments in the Internet industry and comment on some limitations of our proposed model.

6.1 Implementation and Regulatory Issues

Subsidization works with the usage-based pricing schemes of the access ISPs. For mobile providers, implementing data caps and accounting for data traffic consumptions are common in practice. Broadband providers in countries like India and Australia have been using usage-based pricing, and major U.S. ISPs like Verizon [43] and AT&T [47] have also adopted usage-based pricing after gaining support from the FCC [42, 22]. To differentiate content and enable subsidization from different CPs, platforms such as FreeBand [3] have been developed to provide data traffic statistics to mobile carriers so as to lower the bills of consumers, based on the type of applications. AT&T's recent sponsored data [1] plan allows CPs to fully subsidize their users, and Syntonic Wireless [7] provides the corresponding *Sponsored Content Store* for end-users to use sponsored applications. Our model allows partial subsidization and the implementation would be easy if the CP has a direct financial relationship with its end-users, e.g., a shopping site. Otherwise, ISPs could implement accounting mechanisms to realize the subsidization as an intermediary; however, they would be preferred not to be involved in the payment transfer if net neutrality is concerned.

Deregulation of subsidization might reduce the throughput of certain CPs, e.g., the upper-left ones in Figure 9. These CPs either do not have incentives to subsidize, because their users are not price-sensitive, i.e., the elasticity $\epsilon^{m_i}_{s_i}$ is small, or they cannot afford to subsidize, because their profitability v_i is low. In the former case, the decrease in throughput is mainly due to a higher utilization of the system, which will be relieved in the long term when the ISPs expand capacities. Startup companies with low profits are of the latter case. The decrease in throughput is mainly due to the ISP's high price that limits the user demand, and therefore, regulators might want to regulate ISP's monopolistic pricing or introduce competition, e.g., municipal infrastructures. However, if they are promising, we also expect that venture capital would provide the funding source for them to subsidize their users and achieve their potential high profitability. After all, we believe in a transparent and competitive market where users can choose CPs based on quality and prices, and the businesses drive the evolution of the Internet ecosystem.

Although our analyses imply that a deregulation of subsidization will provide CPs more freedom to subsidize and increase the competitiveness of the content market, regulators might still want to impose policies that prevent discriminatory subsidization of the access ISPs. For example, Comcast's Xbox XFinity [2] service enables its users to access on-demand content on their Xbox devices, which does not count against their 250GB data cap. By implicitly subsidizing their users, this service favors the ISP's vertically integrated business over other content competitors, e.g., Netflix. Some

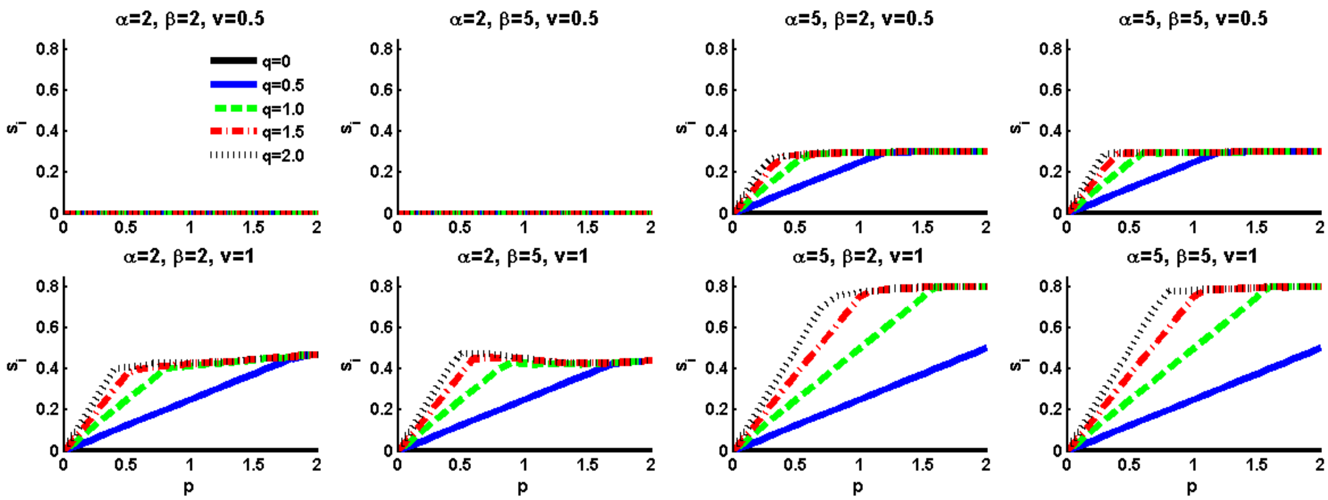

Figure 8: The subsidies s_i of the 8 types of CPs under equilibrium.

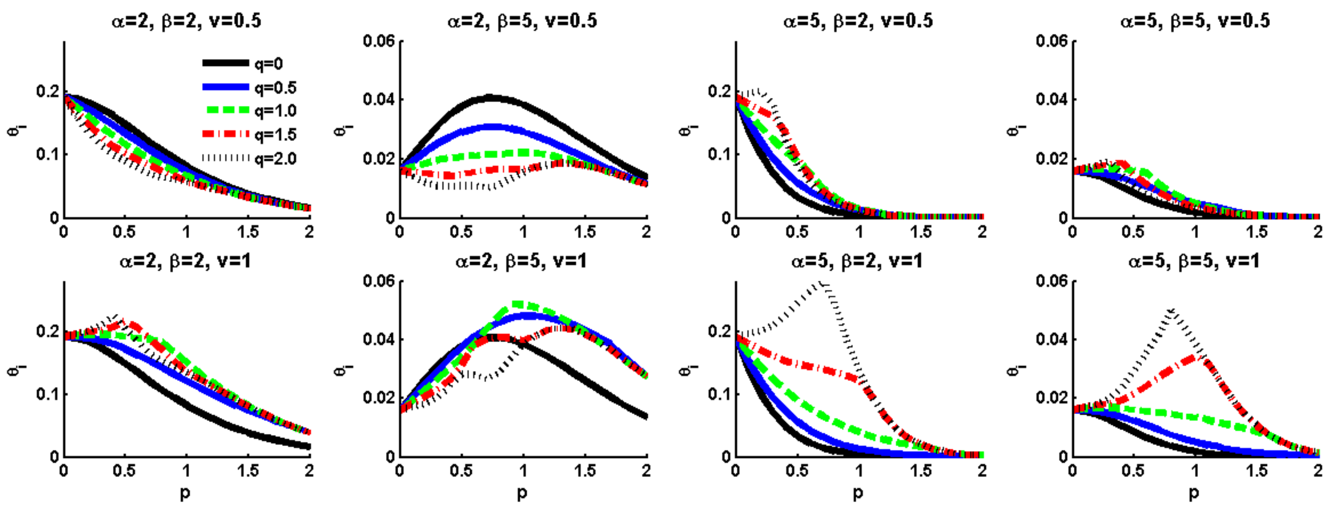

Figure 9: The throughput θ_i of the 8 types of CPs under equilibrium.

CPs work with access ISPs to provide better services and cheaper prices for their users, e.g., Google Apps for Verizon [4]; however, this might also inversely affect the competitors of those favored CPs. We believe that the subsidization option should be identical, and should be given to all CPs equally. Through this, the access ISPs would treat CPs more neutrally and the subsidization would create a transparent and uniform platform for the CPs to compete.

6.2 Connections to Industrial Developments

Our proposed subsidization framework is mostly close to AT&T's sponsored data [47], under which CPs could pay the access ISP so as to fully subsidize their users. However, there are two major differences. First, our framework is more flexible such that partial subsidization is allowed and determined by CPs. Second, our framework is more fair and transparent, i.e., the usage-based pricing for users are the same for all CPs and the subsidizations paid by CPs are equally treated. Much efforts have been made by CPs to provide better services for their users, e.g., Netflix's OpenConnect [6] and Google Global Cache (GGC) [5]. These efforts make the Internet transit more efficient and competitive. Nevertheless, to support quality of service for content delivery, more severe issues are related to the monopolistic position of the last-mile broadband providers and the scarce capacity of the wireless providers. This is why Netflix has agreed to pay access ISPs like Comcast [40] and Verizon [19] via paid-peering agreements for better content delivery. Although paid-peering provides higher revenue for the access ISPs and some CPs do have incentives to pay for it, access ISPs need to actively manage their network capacity and differentiate CPs' traffic, in other words, maintaining a non-neutral physical network. Our proposed subsidization framework, on the other hand, is a pricing mechanism, which preserves the neutrality of the physical network.

6.3 Model Limitations

Our analytical framework is based on a macroscopic equilibrium model that captures the interactions among an access ISP, a set of CPs and end-users via the CPs' subsidization and the ISP's capacity and pricing decisions. Our theoretical results are mostly qualitative, which provides understanding and predictions of the changes in the system states when some of the driving factors change. The limitation is that it might not be able to capture short-term off-equilibrium types of system dynamics, where players' decisions are not rational or optimal. Our numerical evaluations are limited to a styled model of CPs to capture qualitative trends; however, more detailed experiments and validations could be challenging, because market data are needed so as to obtain the characteristics of the CPs, e.g., their profitability and elasticities, and an ISP needs to execute such a plan in the market. With the emerging sponsored data plan from AT&T, we expect this type of market data could be available for regulatory authorities as well as research communities. This study focuses on a single access ISP; however, we believe that competition between ISPs will also incentivize them to adopt subsidization schemes, through which users can obtain subsidized services. Finally, our result shows that under the subsidization mechanism, the access ISP could increase utilization and revenue. Because ISPs care about their profit, which is a function of both utilization and revenue, the increase in profit will further provide investment incentives for them to expand capacity. We do not have sufficient space to study the ISP's capacity planning decision in detail, which is a direction of our future work.

7. CONCLUSIONS

The Internet is facing a dilemma of increasing traffic demand and decreasing incentives for the access ISPs to expand capacity, because service prioritization and two-sided pricing are still under debate in terms of their implications for net neutrality. We propose to allow CPs to voluntarily subsidize the data usage costs incurred at the access ISPs for their end-users. Our solution modularizes the issue along the tussle boundary such that the physical network is kept neutral. Through modeling and analyses, we show that a deregulated subsidization policy will incentivize profitable CPs to subsidize users and increase the system welfare and the revenue of the access ISPs. With the improved profit margins, the access ISPs would obtain more investment incentives under subsidization. However, the deregulation of subsidization might trigger an ISP to increase price, which would decrease the system welfare as well as the throughput of certain CPs. We suggest that price regulation might be needed if the access market is not competitive enough and the price is too high. Because subsidization creates a feedback channel for CPs to return value to the access ISPs and realign the created value and investment in the value chain, we believe that subsidization competition could vitalize neutral Internet for the future.

8. ACKNOWLEDGMENTS

The author would like to thank the shepherd of this paper, Matthew Andrews, for useful advices on revising this work. The author would also like to thank Vishal Misra for discussions and suggestions, and the anonymous reviewers for their insightful comments.

This study is supported by Ministry of Education of Singapore AcRF grant R-252-000-526-112 and the research grant for the Human Sixth Sense Programme at the Advanced Digital Sciences Center from Singapore's Agency for Science, Technology and Research (A*STAR).

APPENDIX

A. PROOFS OF SELECTED RESULTS

Proof of Lemma 1: By Assumption 1, we know $\Theta(\phi,\mu)$ is strictly increasing in ϕ and $\lambda_i(\phi)$ is strictly decreasing in ϕ; therefore, $g(\phi) = \Theta(\phi,\mu) - \sum_{i\in\mathcal{N}} m_i\lambda_i(\phi)$ is strictly increasing in ϕ. Because $\lim_{\phi\to 0}\Theta(\phi,\mu) = 0$ and $\lambda_i(0) > 0$ for all $i\in\mathcal{N}$, $lim_{\phi\to 0} g(\phi) < 0$; because $\lim_{\phi\to\infty}\lambda_i(\phi) = 0$ for all $i\in\mathcal{N}$, $lim_{\phi\to\infty} g(\phi) = \lim_{\phi\to\infty}\Theta(\phi,\mu) > 0$. As a result, $g(\phi) = 0$ must have a unique solution.

Finally, when $g(\phi) = 0$, $\Theta(\phi,\mu) = \sum_{i\in\mathcal{N}} m_i\lambda_i(\phi)$, which is equivalent to Equation (1) of Definition 1.

■

Proof of Theorem 1: Because $g(\phi)$ is strictly increasing in ϕ by Lemma 1 and by Equation (2), we have

$$\frac{dg}{d\phi} = \frac{\partial\Theta}{\partial\phi} - \sum_{k\in\mathcal{N}} m_k \frac{d\lambda_k}{d\phi} > 0.$$

When considering the capacity effect, we write the system utilization $\phi(\mu)$ as a function of the system capacity. By Lemma 1, $g(\phi(\mu)) = \Theta(\phi(\mu),\mu) - \sum_{k\in\mathcal{N}} m_k\lambda_k(\phi(\mu)) = 0$. By taking derivative of μ on both sides, we obtain

$$\frac{\partial\Theta}{\partial\phi}\frac{\partial\phi}{\partial\mu} + \frac{\partial\Theta}{\partial\mu} - \sum_{k\in\mathcal{N}} m_k \frac{d\lambda_k}{d\phi}\frac{\partial\phi}{\partial\mu} = 0,$$

which is equivalent to

$$\left(\frac{\partial\Theta}{\partial\phi} - \sum_{k\in\mathcal{N}} m_k \frac{d\lambda_k}{d\phi}\right)\frac{\partial\phi}{\partial\mu} = -\frac{\partial\Theta}{\partial\mu} \quad \text{or} \quad \frac{dg}{d\phi}\frac{\partial\phi}{\partial\mu} = -\frac{\partial\Theta}{\partial\mu}.$$

Since $\Theta(\phi,\mu)$ is increasing in μ, $\partial\Theta/\partial\mu > 0$, which implies

$$\frac{\partial\phi}{\partial\mu} = -\left(\frac{dg}{d\phi}\right)^{-1}\frac{\partial\Theta}{\partial\mu} < 0.$$

When considering the user effect, we write the system utilization $\phi(\mathbf{m})$ as a function of the user populations. By Lemma 1, $g(\phi(\mathbf{m})) = \Theta(\phi(\mathbf{m}),\mu) - \sum_{k\in\mathcal{N}} m_k\lambda_k(\phi(\mathbf{m})) = 0$. By taking derivative of m_i on both sides, we obtain

$$\frac{\partial\Theta}{\partial\phi}\frac{\partial\phi}{\partial m_i} - \sum_{k\in\mathcal{N}} m_k \frac{d\lambda_k}{d\phi}\frac{\partial\phi}{\partial m_i} - \lambda_i = 0,$$

which is equivalent to

$$\left(\frac{\partial\Theta}{\partial\phi} - \sum_{k\in\mathcal{N}} m_k \frac{d\lambda_k}{d\phi}\right)\frac{\partial\phi}{\partial m_i} = \lambda_i \quad \text{or} \quad \frac{dg}{d\phi}\frac{\partial\phi}{\partial m_i} = \lambda_i.$$

Since $\frac{dg}{d\phi} > 0$, the above implies $\frac{\partial\phi}{\partial m_i} = \left(\frac{dg}{d\phi}\right)^{-1}\lambda_i > 0$. Finally, the results for $\partial\theta_i/\partial\mu$, $\partial\theta_i/\partial m_i$ and $\partial\theta_j/\partial m_i$ can be derived by using $\theta_i(\phi) = m_i\lambda_i(\phi)$ and chain rules, and applying the above results for $\partial\phi/\partial\mu$ and $\partial\phi/\partial m_i$.

■

Proof of Lemma 3: Because $s'_i > s_i$, $t'_i = p - s'_i < p - s_i = t_i$ and by Assumption 2, $m'_i = m_i(t'_i) \geq m_i(t_i) = m_i$.

By Theorem 1, we know that $\partial\phi/\partial m_i > 0$, and therefore, $\phi(\mathbf{s}') \geq \phi(\mathbf{s})$. Also by Theorem 1, we know that $\partial\theta_i/\partial m_i > 0$ and $\partial\theta_j/\partial m_i < 0$ for all $j \neq i$. As a result, $\theta_i(\mathbf{s}') \geq \theta_i(\mathbf{s})$ and $\theta_j(\mathbf{s}') \leq \theta_j(\mathbf{s})$ for all $j \neq i$, which implies $U_j(\mathbf{s}') = v_j\theta_j(\mathbf{s}') \leq v_j\theta_j(\mathbf{s}) = U_j(\mathbf{s})$ for all $j \neq i$.

■

Proof of Theorem 3: If $\mathbf{s}$ is a Nash equilibrium, each $s_i \in [0,q]$ maximizes $U_i(s_i;\mathbf{s}_{-i}) = (v_i - s_i)m_i(p - s_i)\lambda_i(\phi(\mathbf{s}))$. By Karush-Kuhn-Tucker condition, we can derive that

$$\frac{\partial U_i}{\partial s_i} \begin{cases} \leq 0 & \text{if } s_i = 0; \\ \geq 0 & \text{if } s_i = q; \\ = 0 & \text{if } 0 < s_i < q; \end{cases} \qquad \forall\, i\in\mathcal{N}. \tag{18}$$

For CP i with $s_i = 0$, the above condition tells that

$$\frac{\partial U_i}{\partial s_i} = (v_i - s_i)\frac{\partial\theta_i}{\partial s_i} - \theta_i = v_i\frac{\partial\theta_i}{\partial s_i} - \theta_i \leq 0.$$

By Lemma 3, we know $\partial\theta_i/\partial s_i > 0$ and the above implies $v_i \le (\partial\theta_i/\partial s_i)^{-1}\theta_i$. By taking partial derivatives of s_i on $U_i = (v_i - s_i)m_i\lambda_i$, we deduce that

$$\frac{\partial U_i}{\partial s_i} = (v_i - s_i)\left(m_i\frac{\partial\lambda_i}{\partial s_i} + \lambda_i\frac{\partial m_i}{\partial s_i}\right) - m_i\lambda_i.$$

Dividing $m_i\lambda_i$ on both sides of (18), we have

$$(v_i - s_i)\left(\frac{1}{\lambda_i}\frac{\partial\lambda_i}{\partial s_i} + \frac{1}{m_i}\frac{\partial m_i}{\partial s_i}\right)\begin{cases} \le 1 & \text{if } s_i = 0; \\ \ge 1 & \text{if } s_i = q; \\ = 1 & \text{if } 0 < s_i < q; \end{cases}$$

When we multiply s_i on both sides of the above, the left hand side becomes

$$(v_i - s_i)\left(\epsilon^{m_i}_{s_i} + \epsilon^{\lambda_i}_{s_i}\right) = (v_i - s_i)\left(\epsilon^{m_i}_{s_i} + \epsilon^{\lambda_i}_{\phi}\epsilon^{\phi}_{m_i}\epsilon^{m_i}_{s_i}\right) = \tau_i(\mathbf{s}).$$

When $s_i = 0$, $\tau_i(\mathbf{s}) = 0 = s_i$, and therefore, we have

$$\tau_i(\mathbf{s})\begin{cases} \ge s_i & \text{if } s_i = q; \\ = s_i & \text{if } 0 \le s_i < q, \end{cases} \quad \text{or} \quad s_i = \min\{\tau_i(\mathbf{s}), q\}.$$

Finally, if $U_i(\mathbf{s})$ is concave in s_i for all $i \in \mathcal{N}$, local optimality also guarantees the global optimality and therefore, the above first order necessary conditions become sufficient condition for $\mathbf{s}$ being a Nash equilibrium. ∎

Proof of Theorem 4: The condition (10) implies that $U_i(s_i; \mathbf{s}_{-i})$ is concave in s_i for any $\mathbf{s}_{-i}$. Because the strategy space $[0, q]$ is compact, it guarantees the existence of Nash equilibrium. Suppose there exists two distinct Nash equilibria $\hat{\mathbf{s}}$ and $\tilde{\mathbf{s}}$. By concavity of $U_i(s_i; \mathbf{s}_{-i})$ in s_i and the maximum principle, for any $i \in \mathcal{N}$ and any $x_i \in [0, q]$, $(x_i - \hat{s}_i)u_i(\hat{\mathbf{s}}) \le 0$ and $(x_i - \tilde{s}_i)u_i(\tilde{\mathbf{s}}) \le 0$. By substituting $\mathbf{x} = \tilde{\mathbf{s}}$ in the first inequality and $\mathbf{x} = \hat{\mathbf{s}}$ in the second inequality, we deduce for any $i \in \mathcal{N}$, $(\tilde{s}_i - \hat{s}_i)u_i(\hat{\mathbf{s}}) \le 0$, and $(\hat{s}_i - \tilde{s}_i)u_i(\tilde{\mathbf{s}}) \le 0$. By adding the above inequalities, we further deduce that $(\tilde{s}_i - \hat{s}_i)(u_i(\hat{\mathbf{s}}) - u_i(\tilde{\mathbf{s}})) \le 0, \quad \forall i \in \mathcal{N}$. The above is equivalent to $(\tilde{s}_i - \hat{s}_i)(u_i(\tilde{\mathbf{s}}) - u_i(\hat{\mathbf{s}})) \ge 0$ for all $i \in \mathcal{N}$, which contradicts the condition (10). ∎

Proof of Theorem 5: For any CP $j \in \mathcal{N}\backslash\{i\}$, we show that $(\hat{s}_j - s_j)(u_j(\hat{\mathbf{s}}) - u_j(\mathbf{s})) \ge 0$. Without loss of generality, we consider the cases of $s_j \le \hat{s}_j$. By the KKT condition of (18), we have 4 cases. 1) $s_j = \hat{s}_j$: we have $(\hat{s}_j - s_j)(u_j(\hat{\mathbf{s}}) - u_j(\mathbf{s})) = 0$. 2) $s_j = 0$ and $\hat{s}_j \in (0, q)$: $u_j(\mathbf{s}) < 0$ and $u_j(\hat{\mathbf{s}}) = 0$; and therefore, the above becomes $\hat{s}_j(-u_j(\mathbf{s})) \ge 0$. 3) $s_j = 0$ and $\hat{s}_j = q$: $u_j(\mathbf{s}) < 0$ and $u_j(\hat{\mathbf{s}}) > 0$; and therefore, we have $q(u_j(\hat{\mathbf{s}}) - u_j(\mathbf{s})) \ge 0$. 4) $s_j \in (0, q)$ and $\hat{s}_j = q$: $u_j(\mathbf{s}) = 0$ and $u_j(\hat{\mathbf{s}}) > 0$; and therefore, the above becomes $(q - s_j)u_j(\hat{\mathbf{s}}) \ge 0$.

By the condition (10), we have $(\hat{s}_i - s_i)(u_i(\hat{\mathbf{s}}) - u_i(\mathbf{s})) < 0$. Suppose $\hat{s}_i < s_i$, the above implies that $u_i(\hat{\mathbf{s}}) > u_i(\mathbf{s})$. However, if $\hat{s}_i < s_i$, we must have $s_i > 0$ and by the KKT condition of (18), we have $u_i(\mathbf{s}) \ge 0$, which further implies that $u_i(\hat{\mathbf{s}}) > u_i(\mathbf{s}) \ge 0$. Again by the KKT condition of (18), $u_i(\hat{\mathbf{s}}) > 0$ implies $\hat{s} = q$ which contradicts with the assumption $\hat{s}_i < s_i$. Consequently, we have $\hat{s}_i \ge s_i$. ∎

Proof of Theorem 6: The condition (10) of Theorem 4 implies the local concavity of the utility functions. By Proposition 1.4.2 of [17], the Nash equilibrium $\mathbf{s}$ can be equivalently characterized as the solution of a variational inequality, denoted as $VI(F, K)$, where $F \triangleq -\mathbf{u}$ and $K \triangleq [0, q]^{|\mathcal{N}|}$. $\mathbf{s} \in \mathcal{K}$ is a solution of $VI(F, K)$ if $(\mathbf{x} - \mathbf{s})^T F(\mathbf{s}) \ge 0$ for all $\mathbf{x} \in K$. We apply the sensitivity analysis [48, 15] of variational inequalities to obtain the dynamics of the Nash equilibrium $\mathbf{s}(p, q)$ as the price p or the policy q changes.

For each CP i, the constraint $s_i \in [0, q]$ can be written as two linear constraints $g^-_i(\mathbf{s}) \triangleq s_i \ge 0$ and $g^+_i(\mathbf{s}) \triangleq q - s_i \ge 0$. For any $\mathbf{s}$, the set of binding constraints are

$$\mathcal{G} = \{g^-_i(\mathbf{s}) : i \in \mathcal{N}_-\} \cup \{g^+_i(\mathbf{s}) : i \in \mathcal{N}_+\}. \tag{19}$$

Because $\mathcal{N}_- \cap \mathcal{N}_+ = \emptyset$, the gradients of the binding constraints are linearly independent. Because $u_i(\mathbf{s}) = 0$ implies $s_i \in \widetilde{\mathcal{N}}$ for all $i \in \mathcal{N}$, by the KKT condition of (18), we know that the strict complementary slackness condition of Theorem 3.1 of [48] holds. By Theorem 4, the Nash equilibrium is locally unique, and therefore, by Theorem 3.1 of [48], in a neighborhood of (p, q), the Nash equilibrium can be written as a differentiable function $\mathbf{s}(p, q)$.

Let $m \triangleq |\mathcal{N}_-| + |\mathcal{N}_+|$ be the number of binding constraints. We define G as a $m \times n$ matrix, whose ith row is the gradient of the ith binding constraint with respect to $\mathbf{s}$. We define E_p and E_q as $m \times 1$ vectors, whose ith component are the partial derivatives of the ith binding constraint with respect to p and q, respectively. Following [15], we define $Q = G^T(GG^T)^{-1}G$ and M to be an $n \times m$ matrix satisfying $MG = Q$ and $QM = M$. Given the set $\mathcal{G}$ of linear constraints in (19), $Q = \{q_{ij}\}$ is a diagonal matrix satisfying $q_{ii} = 1$ if $i \notin \widetilde{\mathcal{N}}$ and $q_{ii} = 0$ otherwise, and $M = G^T$.

When q is the sensitivity parameter, $E_q = \{e_j\}$ satisfies that $e_j = 1$ if the jth constraint is a type of $g^+_i(\mathbf{s}) = q - s_i$ constraint and $e_j = 0$ otherwise. Because the constraints $\mathcal{G}$ are linear, Theorem 3.1 of [15] implies that $Q\nabla_q\mathbf{s}(q) = -ME_q$ and $(I - Q)(\nabla_\mathbf{s}F\nabla_q\mathbf{s}(q) + \nabla_q F) = \mathbf{0}$. Because when multiplying $\nabla_q\mathbf{s}(q)$ by Q, the values associated with the non-binding strategies vanish, the first equation implies

$$\partial s_i/\partial q = 0, \ \forall i \in \mathcal{N}_- \ \text{ and } \ \partial s_i/\partial q = 1, \ \forall i \in \mathcal{N}_+. \tag{20}$$

Because $F \triangleq -\mathbf{u}$ and $\mathbf{u}$ does not depend on q, the second equation can be written as $(I - Q)\nabla_\mathbf{s}\mathbf{u}\nabla_q\mathbf{s} = \mathbf{0}$. Because when multiplied by $I - Q$, the components associated with the binding strategies vanish, by substituting (20) into the above, we obtain $\nabla_{\tilde{\mathbf{s}}}\tilde{\mathbf{u}}\nabla_q\tilde{\mathbf{s}} + \nabla_\mathbf{s}\tilde{\mathbf{u}}E_q = \mathbf{0}$. By the condition (10) of Theorem 4, we deduce that $F \triangleq -\mathbf{u}$ is a *P-function* [31]. If we restrict to the CPs in $\widetilde{\mathcal{N}}$, we know that the Jacobian $\nabla_{\tilde{\mathbf{s}}}(-\tilde{\mathbf{u}})$ is a *P-matrix* [31], which is always non-singular. Therefore, we deduce $\nabla_q\tilde{\mathbf{s}} = -(\nabla_{\tilde{\mathbf{s}}}\tilde{\mathbf{u}})^{-1}\nabla_\mathbf{s}\tilde{\mathbf{u}}E_q = -\Psi\nabla_\mathbf{s}\tilde{\mathbf{u}}E_q$, the same as the third case of (11).

When taking p as the sensitivity parameter and applying Theorem 3.1 of [15], we can obtain $Q\nabla_p\mathbf{s}(p) = -ME_p$ and $(I - Q)(\nabla_\mathbf{s}F\nabla_p\mathbf{s}(p) + \nabla_p F) = \mathbf{0}$. Because the constraints in $\mathcal{G}$ do not depend on p, E_p is a zero vector, therefore, the first equation implies that $\partial s_i/\partial p = 0$ for all $i \notin \widetilde{\mathcal{N}}$. By substituting $F \triangleq -\mathbf{u}$ into the second equation, we have $(I - Q)(\nabla_\mathbf{s}\mathbf{u}\nabla_p\mathbf{s} + \nabla_p\mathbf{u}) = \mathbf{0}$. Similarly, by substituting $\partial s_i/\partial p = 0$ for all $i \notin \widetilde{\mathcal{N}}$ into the above, we obtain $\nabla_{\tilde{\mathbf{s}}}\tilde{\mathbf{u}}\nabla_p\tilde{\mathbf{s}} + \nabla_p\tilde{\mathbf{u}} = \mathbf{0}$. Because $\nabla_{\tilde{\mathbf{s}}}(-\tilde{\mathbf{u}})$ is non-singular, it is equivalent to the second case of (12) in a matrix form as

$$\nabla_p\tilde{\mathbf{s}} = -(\nabla_{\tilde{\mathbf{s}}}\tilde{\mathbf{u}})^{-1}\nabla_p\tilde{\mathbf{u}} = -\Psi\nabla_p\tilde{\mathbf{u}}. \quad ∎$$

Proof of Corollary 1: By (11) of Theorem 6, $\partial s_i/\partial q \ge 0$ is immediate for $i \notin \widetilde{\mathcal{N}}$. For the CPs in $\widetilde{\mathcal{N}}$, by Theorem 6, we have $\nabla_q\tilde{\mathbf{s}} = -(\nabla_{\tilde{\mathbf{s}}}\tilde{\mathbf{u}})^{-1}\nabla_\mathbf{s}\tilde{\mathbf{u}}E_q = -\Psi\nabla_\mathbf{s}\tilde{\mathbf{u}}E_q$. Because $\mathbf{u}$ is off-diagonally monotone, we know that $\nabla_{\tilde{\mathbf{s}}}(-\tilde{\mathbf{u}})$ is a P-matrix with non-positive off-diagonal entries or an M-matrix [39], which implies that all the entries of $-\Psi$ are nonnegative. By Theorem 6, we know that $\nabla_\mathbf{s}\tilde{\mathbf{u}}E_q$ is a vector of $\{\partial u_i/\partial s_j : i \in \widetilde{\mathcal{N}}, j \in \mathcal{N}_+\}$. Because $\mathbf{u}$ is off-diagonally monotone, the components of $\nabla_\mathbf{s}\tilde{\mathbf{u}}E_q$ are all non-negative, and therefore, $\nabla_q\tilde{\mathbf{s}} = -\Psi\nabla_\mathbf{s}\tilde{\mathbf{u}}E_q \ge \mathbf{0}$.

Now we have shown that $\partial s_i/\partial q \ge 0$ for all $i \in \mathcal{N}$. Since all the subsidies are monotonic, by Lemma 3, we deduce that $\partial\phi/\partial q \ge 0$. Finally, because the ISP's revenue can be written as $R = p\Theta(\phi, \mu)$

and $\partial\Theta(\phi,\mu)/\partial\phi \geq 0$, we have

$$\frac{\partial R}{\partial q} = \frac{\partial R}{\partial \phi}\frac{\partial \phi}{\partial q} = p\frac{\partial\Theta(\phi,\mu)}{\partial\phi}\frac{\partial \phi}{\partial q} \geq 0. \quad \blacksquare$$

B. REFERENCES

[1] AT&T Sponsored Data. http://www.att.com/att/sponsoreddata/.
[2] Comcast Xbox Xfinity. http://xbox.comcast.net/.
[3] Freeband Technologies, Inc. http://www.freebandtechnologies.com.
[4] Google Apps for Verizon. http://www.verizon.com/smallbusiness/products/business-apps/google-apps/.
[5] Google global cache.
[6] Netflix's openconnect.
[7] Syntonic Wireless. http://www.syntonic.com/.
[8] M. Armstrong. Competition in two-sided markets. *RAND Journal of Economics*, 37(3):668–691, 2006.
[9] M. Campbell and J. Browning. Apple, Google asked to pay up as mobile operators face data flood. *Bloomberg News*, December 7 2010.
[10] J. P. Choi and B.-C. Kim. Net neutrality and investment incentives. *The Rand Journal of Economics*, 41(3):446–471, Autumn 2010.
[11] D. D. Clark, J. Wroclawski, K. R. Sollins, and R. Braden. Tussle in cyberspace: defining tomorrow's Internet. *IEEE/ACM Transactions on Networking*, 13(3), June 2005.
[12] C. Courcoubetis, F. Kelly, V. Siris, and R. Weber. A study of simple usage-based charging schemes for broadband networks. *Telecommunication Systems*, 15:323–343, 2000.
[13] C. Courcoubetis and R. Weber. *Pricing Communication Networks: Economics, Technology and Modelling*. John Wiley & Sons Ltd., 2003.
[14] J. Crowcroft. Net neutrality: the technical side of the debate: a white paper. *ACM SIGCOMM Computer Communication Review*, 37(1), January 2007.
[15] S. Dafermos. Sensitivity analysis in variational inequalities. *Mathematics of Operations Research*, 13(3):421–434, August 1988.
[16] N. Economides and J. Tag. Network neutrality and network management regulation: Quality of service, price discrimination, and exclusive contracts. *Research Handbook on Governance of the Internet. Ed. Ian Brown. London: Edward Elgar*, 2012.
[17] F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Ineqaulities and Complementarity Problems*. Springer-Verlag New York, Inc, 2003.
[18] D. Gale and H. Nikaido. The Jacobian matrix and global univalence of mappings. *Mathematische Annalen*, 159(2):81–93, 1965.
[19] S. Gustin. Netflix pays verizon in streaming deal, following comcast pact. *Time Magazine*, Apr. 28 2014.
[20] D. Hawkins and H. A. Simon. Note: Some conditions of macroeconomic stability. *Econometrica*, 17(3/4), 1949.
[21] M. L. Honig and K. Steiglitz. Usage-based pricing of packet data generated by a heterogeneous user population. *Proceedings of IEEE INFOCOM*, 1995.
[22] T. Johnson. FCC chairman supports net neutrality rules but not limits on usage-based pricing. *Variety.com*, (December 2nd), 2013.
[23] G. Kesidis, A. Das, and G. de Veciana. On flat-rate and usage-based pricing for tiered commodity internet services. *Proceedings of the 42nd Annual Conference on Information Sciences and Systems (CISS)*, 2008.
[24] C. Labovitz, D. McPherson, S. Iekel-Johnson, J. Oberheide, and F. Jahanian. Internet inter-domain traffic. In *Proceedings of the ACM SigComm*, New Delhi, India, 2010.
[25] P. Laskowski and J. Chuang. Network monitors and contracting systems: competition and innovation. In *Proceedings of the ACM SigComm*, Pisa, Italy, 2006.
[26] R. S. Lee and T. Wu. Subsidizing creativity through network design: Zero-pricing and net neutrality. *Journal of Economic Perspectives*, 23(3):61–76, 2009.
[27] R. T. B. Ma. Subsidization competition: Vitalizing the neutral Internet. http://arxiv.org/pdf/1406.2516v1.pdf.
[28] R. T. B. Ma, D. Chiu, J. C. Lui, V. Misra, and D. Rubenstein. Internet Economics: The use of Shapley value for ISP settlement. *IEEE/ACM Transactions on Networking*, 18(3), June 2010.
[29] R. T. B. Ma, D. Chiu, J. C. Lui, V. Misra, and D. Rubenstein. On cooperative settlement between content, transit and eyeball Internet service providers. *IEEE/ACM Transactions on Networking*, 19(3), 2011.
[30] J. K. MacKie-Mason and H. R. Varian. Some FAQs about usage-based pricing. *Computer Networks and ISDN Systems*, 28:257–265, 1995.
[31] J. Moré and W. Rheinboldt. On P- and S-functions and related classes of n-dimensional nonlinear mappings. *Linear Algebra and its Applications*, 6:45–68, 1973.
[32] M. Morgan. Pricing schemes key in LTE future. *Telecomasia.net, September 12*, 2011.
[33] J. Musacchio, G. Schwartz, and J. Walrand. A two-sided market analysis of provider investment incentives. *Review of Network Economics*, 8(1):22–39, March 2009.
[34] G. Nagesh. FCC says 'ready to intervene' over AT&T data plan. *The Wall Street Journal, January 8*, 2014.
[35] P. Njoroge, A. E. Ozdaglar, N. E. Stier-Moses, and G. Y. Weintraub. Investment in two sided markets and the net neutrality debate. *Columbia Business School DRO Working Paper No. 2010-05.*, 2010.
[36] W. Norton. The Internet peering playbook: connecting to the core of the Internet. *DrPeering Press*, 2011.
[37] A. Odlyzko. Internet pricing and the history of communications. *Computer Networks*, 36(5):493–517, 2001.
[38] M. Page, L. Rossi, and C. Rand. A viable future model for the Internet. *A. T. Kearney Paper*, December 2010.
[39] G. Poole and T. Boullion. A survey on m-matrices. *SIAM Review*, 16(4):419–427, 1974.
[40] S. Ramachandran. Netflix to pay Comcast for smoother streaming. *The Wall Street Journal*, Feb. 23 2014.
[41] J.-C. Rochet and J. Tirole. Platform competition in two-sided markets. *Journal of the European Economic Association*, 1(4):990–1029, 2003.
[42] A. Schatz and S. E. Ante. FCC chief backs usage-based broadband pricing. *Wall Street Journal*, (December 2), 2010.
[43] L. Segall. Verizon ends unlimited data plan. *CNN Money, July 6*, 2011.
[44] S. Shakkottai, R. Srikant, A. Ozdaglar, and D. Acemoglu. The price of simplicity. *IEEE Journal on Selected Areas of Communications*, 26(7), September 2008.
[45] N. Shetty, G. Schwartz, and J. Walrand. Internet QoS and regulations. *IEEE/ACM Transactions on Networking*, 18(6), December 2010.
[46] J. G. Sidak. A consumer-welfare approach to network neutrality regulation of the Internet. *Journal of Competition Law and Economics*, 2(3):349–474, September 2006.
[47] P. Taylor. AT&T imposes usage caps on fixed-line broadband. *Financial Times, March 14*, 2011.
[48] R. L. Tobin. Sensitivity analysis for variational inequalities. *Journal of Optimization Theory and Applications*, 48(1):191–204, 1986.
[49] T. Wu. Network neutrality, broadband discrimination. *Journal of Telecommunications and High Technology Law*, 2:141–179, 2003.